Regular Totally Separable Sphere Packings

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Abstract

The topic of totally separable sphere packings is surveyed with a focus on regular constructions, uniform tilings, and contact number problems. An enumeration of all regular totally separable sphere packings in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$ which are based on convex uniform tessellations, honeycombs, and tetracombs, respectively, is presented, as well as a construction of a family of regular totally separable sphere packings in $\mathbb{R}^d$ that is not based on a convex uniform d-honeycomb for $d \geq 3$.

Keywords: sphere packings, hyperplane arrangements, contact numbers, separability.
MSC 2010 Subject Classifications: Primary 52B20, Secondary 14H52.

1 Introduction

In the 1940s, P. Erdös introduced the notion of a separable set of domains in the plane, which gained the attention of H. Hadwiger in [1]. G.F. Tóth and L.F. Tóth extended this notion to totally separable domains and proved the densest totally separable arrangement of congruent copies of a domain is given by a lattice packing of the domains generated by the side-vectors of a parallelogram of least area containing a domain [2]. Totally separable domains are also mentioned by G. Kertész in [3], where it is proved that a cube of volume V contains a totally separable set of N balls of radius r with $V \geq 8Nr^3$. Further results and references regarding separability can be found in a manuscript of J. Pach and G. Tardos [4].

This manuscript continues the study of separability in the context of regular unit sphere packings, i.e., infinite sets of unit spheres

$$\mathcal{P} = \bigcup_{i=1}^{\infty} \left(x_i + \mathbb{S}^{d-1}\right)$$

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in $\mathbb{R}^d$ with $\|x_i - x_j\| \geq 2$, whose contact graphs $G_{\mathcal{P}} = (V, E)$, where $V = \{x_i \mid i \in \mathbb{N}\}$ and

$$E = \left\{(i,j) \mid \left(x_i + \mathbb{S}^{d-1}\right) \cap \left(x_j + \mathbb{S}^{d-1}\right) \neq \emptyset\right\},$$

are regular (every vertex has equal degree); this means that every sphere in the packing touches the same number of spheres.

Let $C(\mathcal{P}_n)$ be the contact number of a unit sphere packing $\mathcal{P}_n$ with n spheres, i.e., the cardinality of the edge set of the contact graph $G_{\mathcal{P}_n}$. Determining the maximum contact number of a unit sphere packing with n spheres is known as the contact number problem. The contact number problem for circle packings in $\mathbb{R}^2$ was solved exactly in 1974 by H. Harborth in [5] to be $\lfloor 3n - \sqrt{12n-3} \rfloor$. Upper and lower bounds on the contact number problem for finite packing of unit balls in $\mathbb{R}^3$ were provided by K. Bezdek and the author in [6] and studied in detail up to $n = 18$ by M. Holmes-Cerfon in [7] improving the lower bounds for some values. Consult [8] and references therein for more information regarding contact numbers of unit sphere packings and arrangements of spheres in higher dimensions.

Definition 1. *A sphere packing $\mathcal{P}$ is totally separable if every tangent hyperplane to a pair of touching spheres has an empty intersection with the interior of all spheres in $\mathcal{P}$.*

The contact number problem for totally separable sphere packings is studied and all regular totally separable sphere packings in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$ based on convex uniform tessellations (classified in an unpublished manuscript of G. Olshevsky [10]) are constructed. Now, let

$$c(n,d) = \max_{\mathrm{sep}(\mathcal{P}_n)=1} C(\mathcal{P}_n),$$

where $\mathrm{sep}(\cdot)$ is a measure on sphere packings called the *separability* of the packing which is defined formally in the appendix; intuitively, the separability of a packing is 0 if the packing is inseparable and 1 if it is totally separable. The theory of minimal area polyominoes developed in [9] is used with Euler's formula to provide a proof of the contact number problem for totally separable circle packings:

$$c(n,2) = \left\lfloor 2(n - \sqrt{n}) \right\rfloor.$$

Furthermore, heuristics are provided for the upper bound on the contact number problem for totally separable sphere packings in $\mathbb{R}^d$ which is based on the number of edges of polyominoes over the cubic d-honeycomb and hence exact when $\sqrt[d]{n} \in \mathbb{N}$:

$$c(n,d) \leq \left\lfloor d\left(n - n^{\frac{d-1}{d}}\right) \right\rfloor.$$

As this manuscript was being prepared, K. Bezdek, B. Szalkai, and I. Szalkai proved the above upper bound on $c(n,d)$ with an ingenious argument involving box-polytopes and the isoperimetric inequality [11]. The paper ends with a construction of a family of regular totally separable sphere packings in $\mathbb{R}^d$ that is not based on a convex uniform tessellation for $d \geq 3$ and an outline of future research directions.

The most basic example of when the condition on a totally separable sphere packing is violated is explained in the form of a lemma for future reference.

Lemma 1. *If the contact graph $G_{\mathcal{P}}$ of a sphere packing $\mathcal{P}$ in $\mathbb{R}^d$ contains a k-simplex for $2 \le k \le d$, then $\mathcal{P}$ is not totally separable.*

Proof. First consider the case where $G_{\mathcal{P}}$ contains a 2-simplex and observe that it violates total separability. For, the tangent line generated by the touching circles associated with an edge e of the 2-simplex intersects the interior of the circle associated with the vertex which is not an endpoint of e. Proceed by induction, observing from the base case $d = 2$ that any k-simplex with $3 \le k \le d$ in $G_{\mathcal{P}}$ violates total separability as that k-simplex contains a 2-simplex somewhere in its flag, thus proving the lemma. □

This lemma will be used extensively for classifying totally separable sphere packings based on convex uniform tesselations of $\mathbb{R}^d$, also known as tilings or honeycombs.

2 Regular Totally Separable Circle Packings in $\mathbb{R}^2$

Regular totally separable circle packings in $\mathbb{R}^2$ which are based on convex uniform tilings are classified by the following theorem.

Theorem 1. *There are exactly 4 convex uniform tilings in $\mathbb{R}^2$ which generate totally separable circle packings:*

1. *$\mathcal{P}1$ - Square tiling,* $\{4, 4, 4\}$
2. *$\mathcal{P}3$ - Hexagonal tiling,* $\{6, 6, 6\}$
3. *$\mathcal{K}6$ - Truncated square tiling,* $\{4, 8, 8\}$
4. *$\mathcal{K}9$ - Omnitruncated trihexagonal tiling,* $\{4, 6, 12\}$

Proof. Apply Lemma 1 to the list of 11 convex uniform tilings of $\mathbb{R}^2$; three Pythagorean tilings and eight Keplerian tilings [10]. Clearly, if $\mathcal{P}$ is a 4-regular totally separable packing of unit circles in $\mathbb{R}^2$ generated by a convex uniform tiling, then $\mathcal{P}$ is congruent to $\mathcal{P}1$. If $\mathcal{P}$ is a 3-regular totally separable packing of unit circles in $\mathbb{R}^2$ generated by a convex uniform tiling, then $\mathcal{P}$ is congruent to $\mathcal{P}3$, $\mathcal{K}6$, $\mathcal{K}9$ or a subset of $\mathcal{P}1$. If $\mathcal{P}$ is a 2-regular totally separable packing of unit circles in $\mathbb{R}^2$ generated by a convex uniform tiling, then $\mathcal{P}$ is congruent to a subset of either $\mathcal{P}1$, $\mathcal{P}3$, $\mathcal{K}6$, or $\mathcal{K}9$. □

The theory of minimal area polyominoes and Euler's formula is used to provide an exact solution to the contact number problem for totally separable circle packings; an alternative explicit proof, not relying on the results of [9], which extends a proof technique of H. Harborth [5] appears in [11].

Theorem 2. *Given $n \in \mathbb{N}$, there exists a totally separable circle packing $\mathcal{P}_n$ in $\mathbb{R}^2$ with contact number*

$$C(\mathcal{P}_n) = \left\lfloor 2(n - \sqrt{n}) \right\rfloor.$$

Furthermore, no totally separable circle packing in $\mathbb{R}^2$ has a larger contact number.

Proof. By Euler's formula, $n - (|E| + P(c)) + a = 2$, where $|E|$ is the cardinality of the edge set of the contact graph $G_{\mathcal{P}_n}$, $P(c)$ is the perimeter of the polyomino c with area a generated by placing n unit 2-cubes so that elements of $\mathcal{P}_n$ are incircles. Interpolate the piece-wise defined function from Corollary 2.5 of [9] which provides the minimal perimeter of a polyomino of area a in order to obtain the desired formula. □

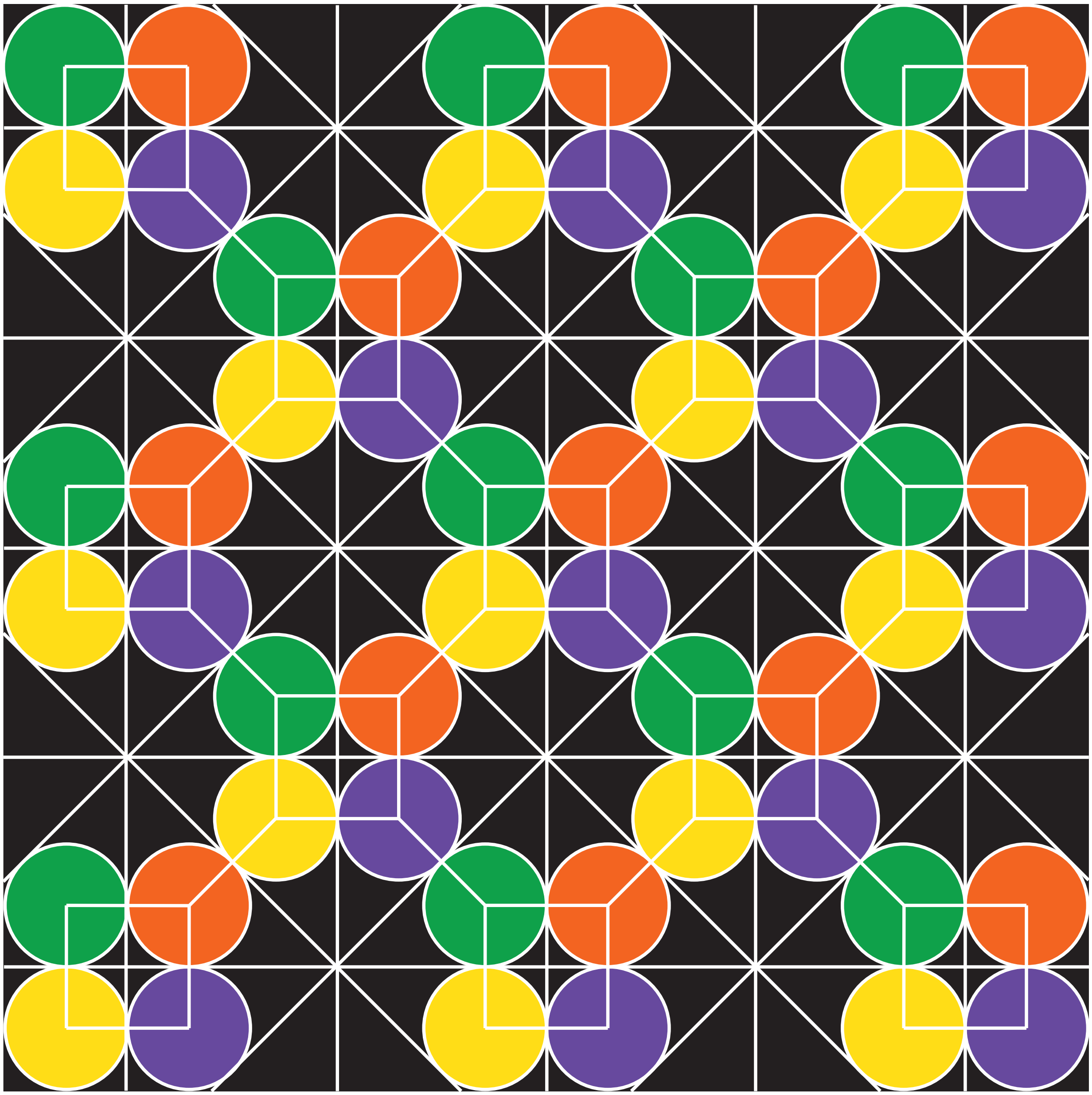

Figure 1: A finite part of the contact graph, convex uniform tiling, and 3-regular totally separable circle packing generated by the truncated square tiling.

3 Regular Totally Separable Sphere Packings in $\mathbb{R}^3$

Regular totally separable sphere packings in $\mathbb{R}^3$ which are based on convex uniform honeycombs are classified by the following theorem.

Theorem 3. *There are exactly 7 convex uniform honeycombs in $\mathbb{R}^3$ which generate totally separable sphere packings in $\mathbb{R}^3$:*

1. *$\mathcal{J}1$ - Cubic honeycomb*
2. *$\mathcal{J}3$ - Hexagonal prismatic honeycomb*
3. *$\mathcal{J}6$ - Truncated square prismatic honeycomb*
4. *$\mathcal{J}9$ - Omnitruncated trihexagonal prismatic honeycomb*
5. *$\mathcal{J}16$ - Bitruncated cubic honeycomb*
6. *$\mathcal{J}18$ - Cantitruncated cubic honeycomb*
7. *$\mathcal{J}20$ - Omnitruncated cubic honeycomb*

Proof. Apply Lemma 1 to N. Johnson's list of 28 convex uniform honeycombs [12]. Clearly, if $\mathcal{P}$ is a 6-regular totally separable packing of unit spheres in $\mathbb{R}^3$ generated by a convex uniform honeycomb, then $\mathcal{P}$ is congruent to $\mathcal{J}1$. If $\mathcal{P}$ is a 5-regular totally separable packing of unit spheres in $\mathbb{R}^3$ generated by a convex uniform honeycomb, then $\mathcal{P}$ is congruent to $\mathcal{J}3$, $\mathcal{J}6$, $\mathcal{J}9$, or a subset of $\mathcal{J}1$. If $\mathcal{P}$ is a 4-regular totally separable packing of unit spheres in $\mathbb{R}^3$ generated by a convex uniform honeycomb, then $\mathcal{P}$ is congruent to $\mathcal{J}16$, $\mathcal{J}18$, $\mathcal{J}20$, or a subset of either $\mathcal{J}1$, $\mathcal{J}3$, $\mathcal{J}6$, or $\mathcal{J}9$. If $\mathcal{P}$ is a 3-regular, or 2-regular totally separable packing of unit spheres in $\mathbb{R}^3$ generated by a convex uniform honeycomb, then $\mathcal{P}$ is congruent to a subset of either $\mathcal{J}1$, $\mathcal{J}3$, $\mathcal{J}6$, $\mathcal{J}9$, $\mathcal{J}16$, $\mathcal{J}18$, or $\mathcal{J}20$. $\square$

4 Regular Totally Separable Sphere Packings in $\mathbb{R}^4$

Regular totally separable sphere packings in $\mathbb{R}^4$ based on convex uniform 4-honeycombs are classified by the following theorem.

Theorem 4. *There are exactly* 18 *convex uniform tetracombs in $\mathbb{R}^4$ which generate totally separable sphere packings in $\mathbb{R}^4$:*

1. *$\mathcal{O}1$ - Tesseractic tetracomb*
2. *$\mathcal{O}3$ - Square-hexagonal duoprismatic tetracomb*
3. *$\mathcal{O}6$ - Tomosquare-square duoprismatic tetracomb*
4. *$\mathcal{O}9$ - Omnitruncated-trihexagonal-square duoprismatic tetracomb*

5. *$\mathcal{O}$16 - Bitruncated-cubic prismatic tetracomb*
6. *$\mathcal{O}$18 - Cantitruncated-cubic prismatic tetracomb*
7. *$\mathcal{O}$20 - Omnitruncated-cubic prismatic tetracomb*
8. *$\mathcal{O}$39 - Hexagonal duoprismatic tetracomb*
9. *$\mathcal{O}$42 - Hexagonal-tomosquare duoprismatic tetracomb*
10. *$\mathcal{O}$45 - Hexagonal-omnitruncated-trihexagonal duoprismatic tetracomb*
11. *$\mathcal{O}$63 - Tomosquare duoprismatic tetracomb*
12. *$\mathcal{O}$66 - Tomosquare-omnitruncated-trihexagonal duoprismatic tetracomb*
13. *$\mathcal{O}$78 - Omnitruncated-trihexagonal duoprismatic tetracomb*
14. *$\mathcal{O}$99 - Truncated icositetrachoric tetracomb*
15. *$\mathcal{O}$100 - Great diprismatotesseractic tetracomb*
16. *$\mathcal{O}$103 - Omnitruncated tesseractic tetracomb*
17. *$\mathcal{O}$132 - Omnitruncated icositetrachoric tetracomb*
18. *$\mathcal{O}$140 - Great-prismatodecachoric tetracomb*

Proof. Apply Lemma 1 to G. Olshevsky's list of 143 convex uniform 4-honeycombs [10]. Clearly, if $\mathcal{P}$ is a 8-regular totally separable packing of unit spheres in $\mathbb{R}^4$ generated by a convex uniform tetracomb, then $\mathcal{P}$ is congruent to $\mathcal{O}1$. If $\mathcal{P}$ is a 7-regular totally separable packing of unit spheres in $\mathbb{R}^4$ generated by a convex uniform tetracomb, then $\mathcal{P}$ is congruent to $\mathcal{O}3$, $\mathcal{O}6$, $\mathcal{O}9$, or a subset of $\mathcal{O}1$. If $\mathcal{P}$ is a 6-regular totally separable packing of unit spheres in $\mathbb{R}^4$ generated by a convex uniform tetracomb, then $\mathcal{P}$ is congruent to $\mathcal{O}16$, $\mathcal{O}18$, $\mathcal{O}20$, $\mathcal{O}39$, $\mathcal{O}42$, $\mathcal{O}45$, $\mathcal{O}63$, $\mathcal{O}66$, $\mathcal{O}78$, or a subset of either $\mathcal{O}1$, $\mathcal{O}3$, $\mathcal{O}6$, or $\mathcal{O}9$. If $\mathcal{P}$ is a 5-regular totally separable packing of unit spheres in $\mathbb{R}^4$ generated by a convex uniform tetracomb, then $\mathcal{P}$ is congruent to $\mathcal{O}99$, $\mathcal{O}100$, $\mathcal{O}103$, $\mathcal{O}132$, $\mathcal{O}140$, or a subset of either $\mathcal{O}1$, $\mathcal{O}3$, $\mathcal{O}6$, $\mathcal{O}9$, $\mathcal{O}16$, $\mathcal{O}18$, $\mathcal{O}20$, $\mathcal{O}39$, $\mathcal{O}42$, $\mathcal{O}45$, $\mathcal{O}63$, $\mathcal{O}66$, or $\mathcal{O}78$. If $\mathcal{P}$ is a 4-regular, 3-regular, or 2-regular totally separable packing of unit spheres in $\mathbb{R}^4$ generated by a convex uniform tetracomb, then $\mathcal{P}$ is congruent to a subset of either $\mathcal{O}1$, $\mathcal{O}3$, $\mathcal{O}6$, $\mathcal{O}9$, $\mathcal{O}16$, $\mathcal{O}18$, $\mathcal{O}20$, $\mathcal{O}39$, $\mathcal{O}42$, $\mathcal{O}45$, $\mathcal{O}63$, $\mathcal{O}66$, $\mathcal{O}78$, $\mathcal{O}99$, $\mathcal{O}100$, $\mathcal{O}103$, $\mathcal{O}132$, or $\mathcal{O}140$. $\square$

The regularity of each 4-honeycomb is determined by inspecting the number of vertices of the vertex figure associated with the honeycomb, e.g., the vertex figure of $\mathcal{O}100$ is an irregular pentachoron, implying that the 4-dimensional sphere packing generated by the great diprismatotessseractic tetracomb is 5-regular.

5 Totally Separable Sphere Packings in $\mathbb{R}^d$

Totally separable sphere packings in $\mathbb{R}^d$ are studied and future research directions are outlined. The following heuristics for the upper bound to the contact number problem for totally separable sphere packings in $\mathbb{R}^d$ provides a reasonable intuitive explanation of the following theorem.

From the formula for the number of m-cubes on the boundary of a d-cube for $m = 1$ observe that

$$2^{d-1}\binom{d}{1} = \left\lfloor d\left(2^d - (2^d)^{\frac{d-1}{d}}\right)\right\rfloor = \left\lfloor d\left(n - n^{\frac{d-1}{d}}\right)\right\rfloor$$

for $n = 2^d$. Similarly, for any $n = \sqrt[d]{k} \in \mathbb{N}$ there is a $\underbrace{k \times k \times \cdots \times k}_{\text{d times}}$ d-cube with $\left\lfloor d\left(k^d - (k^d)^{\frac{d-1}{d}}\right)\right\rfloor$ edges, implying that the upper bound in the following theorem is an equality. Assume that $k^d < n < (k+1)^d$ and observe that the upper bound on $c(n,d)$ overestimates the supremum over edge cardinalities of $(k+\delta_1) \times (k+\delta_2) \times \cdots \times (k+\delta_d)$ unit polyominoes with n cells, where $\delta_i \in \{0,1\}$.

Theorem 5. *For $n \in \mathbb{N}$,*

$$c(n,d) \leq \left\lfloor d\left(n - n^{\frac{d-1}{d}}\right)\right\rfloor,$$

with equality when $\sqrt[d]{n} \in \mathbb{N}$.

Proof. Improving upon an earlier and lengthier unpublished case analytic proof, K. Bezdek, B. Szalkai, and I. Szalkai provide an elegant proof using box-polytopes and the isoperimetric inequality [11]. □

The classification of uniform d-honeycombs is incomplete, leading to great difficulty in establishing the above characterizations of totally separable sphere packings in $d = 2, 3, 4$ for $d \geq 5$. The ongoing work by J. Bowers, G. Olshevsky, N. Johnson, and others of classifying uniform polyterons will soon result in the complete classification of uniform 5-honeycombs, and the study of uniform polypetons generating uniform 6-honeycombs has only recently begun. For $d \geq 7$ there appears to be no significant work on uniform honeycombs; although R. Klitzing has classified certain uniform polytopes up to $d = 8$ [13]. Future research on the topic of regular totally separable sphere packings should include a comprehensive construction of families of k-regular totally separable sphere packings in $\mathbb{R}^d$ for $3 \leq k \leq 2d-1$ and $d \geq 5$. These are the unknown bounds on k-regularity because for $k = 2$ spheres can be placed along an apeirogon (infinite line with evenly spaced points) and for $k = 2d$ spheres can be placed on the cubic d-honeycomb. For an example to motivate future research in this direction, a construction in $\mathbb{R}^d$ of a $(d+1)$-regular totally separable sphere packing which is not based on a convex uniform d-honeycomb for $d \geq 3$ is presented. A similar construction would be desired for $3 \leq k \leq d$ and $d + 2 \leq k \leq 2d - 1$; regardless of whether or not it is based on a convex uniform d-honeycomb.

Theorem 6. *There exists a $(d+1)$-regular totally separable sphere packing in $\mathbb{R}^d$ for $d \geq 3$ which is not based on a convex uniform d-honeycomb.*

Proof. Let $Q_0^d = \operatorname{conv}\left\{x_{0,1}, ..., x_{0,2^d}\right\}$ be a unit d-cube in $\mathbb{R}^d$ and place 2^d unit d-cubes

$$\begin{aligned} Q_1^d &= \operatorname{conv}\left\{x_{1,1}, ..., x_{1,2^d}\right\} \\ &\vdots \\ Q_{2^d}^d &= \operatorname{conv}\left\{x_{2^d,1}, ..., x_{2^d,2^d}\right\} \end{aligned}$$

so that $\|x_{0,1} - x_{1,1}\| = 2, ..., \|x_{0,2^d} - x_{2^d,1}\| = 2$ with $x_{i,1}$ lying outside Q_0^d along a line emanating from the centroid of Q_0^d through $x_{0,i}$ for $1 \leq i \leq 2^d$. Now construct

$$\mathcal{P}_{2^d+4^d} = \bigcup_{i=1}^{2^d+4^d} \bigcup_{j=1}^{2^d} \left(x_{i,j} + \mathbb{S}^{d-1}\right)$$

and iteratively place $2^d - 1$ unit d-cubes diagonally out of each existing unit d-cube $Q_1^d, ..., Q_{2^d}^d$ as above so that spheres may be placed around their vertices which generate a packing congruent to $\mathcal{P}_{2^d+4^d}$. Indefinitely extending this procedure leads to an infinite totally separable sphere packing which is $(d+1)$-regular. For, let $x + \mathbb{S}^{d-1}$ be an arbitrary sphere in this packing and observe that it touches d other spheres placed on adjacent vertices of the unit d-cube which x is a vertex of, and also touches 1 other sphere which is diagonally outward as in the construction. Furthermore, for $d = 2$ this construction corresponds to the truncated square tiling $\mathcal{K}6$ and for $d \geq 3$ this construction corresponds to a scaliform which contains an elongated cubic bifrustum. □

The classification of regular totally separable sphere packings which are not based on convex uniform 3-honeycombs is then a sub-problem of classifying all scaliforms (vertex-transitive honeycombs) in $\mathbb{R}^3$; from a simplex-free scaliform in $\mathbb{R}^3$ one can construct a totally separable sphere packing by placing equal size spheres at the vertices. The questionable existence of aperiodic totally separable sphere packings in any dimension remains unexplored.

Conjecture 1. *No aperiodic totally separable sphere packing exists in any dimension.*

Appendix: Separability as a Geometric Measure

Separability is introduced as a geometric measure where inseparable sphere packings have a separability of 0 and totally separable sphere packings have a separability of 1. Let H_e denote the tangent hyperplane to a pair of touching spheres in $\mathbb{R}^d$ associated with edge e of the contact graph $G_{\mathcal{P}} = (V, E)$. First define the separability measure for finite sphere packings $\mathcal{P}_n$ with $G_{\mathcal{P}_n} = (V_n, E_n)$ by

$$\operatorname{sep}(\mathcal{P}_n) = \sum_{e \in E_n} \frac{\left|\left\{H_e \mid H_e \cap \operatorname{int}\left(x_i + \mathbb{S}^{d-1}\right) = \emptyset, 1 \leq i \leq n\right\}\right|}{|E_n|}.$$

If a sphere packing $\mathcal{P} \hookrightarrow \mathbb{R}^d$ can be constructed so that $\mathcal{P} = \lim_{n\to\infty} \mathcal{P}_n$ for some sequence of finite sphere packings $\mathcal{P}_n$, then

$$\operatorname{sep}(\mathcal{P}) = \lim_{n\to\infty} \sum_{e\in E_n} \frac{\left|\left\{H_e \mid H_e \cap \operatorname{int}\left(x_i + \mathbb{S}^{d-1}\right) = \emptyset, 1 \le i \le n\right\}\right|}{|E_n|}.$$

Observe that if every tangent hyperplane H_e at a contact point associated with the edge e intersects the interior of another sphere in the packing $\mathcal{P}$ then $\operatorname{sep}(\mathcal{P}) = 0$ and similarly if none intersect the interior of a sphere in the packing then $\operatorname{sep}(\mathcal{P}) = 1$; in the former case $\mathcal{P}$ is called inseparable and in the latter case $\mathcal{P}$ is called totally separable.

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References

[1] H. Hadwiger. Nonseparable convex systems. Amer. Math. Monthly: 1947, vol: 54, pp: 583 - 585.

[2] G.F. Tóth. On totally separable domains. Acta mathematica Hungarica: 1973, vol: 24, pp: 229 - 232.

[3] G. Kertész. On totally separable packings of equal balls. Acta mathematica Hungarica: 1988, vol: 51, pp: 363 - 364.

[4] J. Pach and G. Tardos. Separating Convex Sets By Straight Lines. Discrete Mathematics: 2001, vol: 24, pp: 427 - 433.

[5] H. Harborth. Lösung zu Problem 664A. Elem. Math. 29 (1974), 14-15.

[6] K. Bezdek and S. Reid. Contact graphs of unit sphere packings revisited Journal of Geometry: April 2013, vol: 104, pp: 57 - 83.

[7] M. Holmes-Cerfon. Enumerating nonlinearly rigid sphere packings. arXiv:1407.3285 (2014).

[8] K. Bezdek. Lectures on Sphere Arrangements - the Discrete Geometric Side. Springer, 2013.

[9] L. Alonso and R. Cerf. The Three Dimensional Polyominoes of Minimal Area. Electronic Journal of Combinatorics: 1996, vol: 3.

[10] G. Olshevsky. Uniform Panoploid Tetracombs. Unpublished manuscript: 2006, members.aol.com/Polycell/uniform.html

[11] K. Bezdek, B. Szalkai, and I. Szalkai. On contact numbers of totally separable unit sphere packings. arXiv:1501.07907 (2015).

[12] N. Johnson. Uniform Polytopes. Unpublished manuscript.

[13] R. Klitzing. http : //bendwavy.org/klitzing/home.htm